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Dynkin diagrams for hyperbolic Kac–Moody algebras

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Abstract. Hyperbolic algebras such as E_{10} are based on Minkowskian root spaces that revert to finite or affine root systems upon the removal of any simple root. In string theory terms, finite Lie, affine Kac–Moody and hyperbolic Kac–Moody algebras are generated by tachyon vertex operators, tachyon plus photon vertex operators, and vertex operators for all mass levels, respectively. The 136 possible hyperbolic Dynkin diagrams between the ranks 3 and 10 are classified and exhibited, completing an earlier enumeration by Kac of the 18 rank 7–10 cases. The rank-2 hyperbolic algebras, infinite in number, have been classified by Lepowsky and Moody.

1. Introduction and motivation

The algebra E_{10} , often mentioned in the context of string theory, belongs to the class of hyperbolic Kac–Moody algebras whose mathematical theory is not at a stage of development comparable to that of their finite Lie or affine Kac–Moody counterparts. What is currently available in the mathematical literature [1–3] on the subject consists of general conditions on the Cartan matrix, the certainty that higher ($m^2 > 0$) string vertex operators will become part of the generators (resulting in an exponential growth in the number of generators, in contrast to the polynomial growth in the affine case) and upper bounds on the degeneracies of root spaces. Our aim in this paper is to take a simple (but one hopes, useful) step towards the characterisation of such algebras by enumerating their Dynkin diagrams (of rank ≥ 3). A hint of the procedure and partial results are already given in [1]; its detailed implementation, however, leads to a non-trivial amount of work and a surprising proliferation of cases. One can say that these are all ‘exceptional’, both in the sense of not belonging to infinite algebra series, and also in the intimate ties they have with the diagrams of the ordinary exceptional Lie and affine algebras. Hence, although finite in total number, they have to be treated and displayed individually. In fact, the reason behind the finite number and the absence of hyperbolic algebras of rank > 10 is essentially the exceptional nature and finite number of their familiar ancestors. That the highest allowed dimension for a hyperbolic root space of signature $(++ \dots +-)$ is the same as that for the superstring stands out as an intriguing fact.

The interest of physicists and mathematicians in Lorentzian Kac–Moody algebras (of which the hyperbolic algebras form a subclass) stems largely from the most promising method to realise them, which is through an algebra of vertex operators with momenta in a Minkowski space. Versions of vertex operators have proved their utility [4] in integrable systems, in boson–fermion conversions, in endowing the

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heterotic string with internal symmetry, in dual resonance models and in the construction of affine Kac–Moody algebras. It is especially the last two contexts that suggest the potential usefulness of Lorentzian Kac–Moody algebras the most strongly. Let us examine the mathematical one first. As shown by Goddard and Olive, one passes from a finite Lie algebra with Euclidean roots (momenta) to an affine one by adding a single new light-like direction to the root space. (The original Frenkel–Kac–Segal construction [5] uses moments of the Euclidean vertex operator; this is essentially a ‘unitary gauge’ approach.) The natural space in which to embed the resulting singular root space with one zero on the diagonal of its metric is a Minkowskian one, with a second light-like root independent of the first. We will see in § 2 that this gives a Lorentzian algebra of exponential growth. (To have a hyperbolic algebra, this last simple root has to be added to the Dynkin diagram in such a way that the removal of *any* simple root from the resulting diagram leads to a known finite or affine Lie algebra.) Apart from being of interest as natural extensions of affine algebras, Lorentzian algebras are also known to be related to exceptional structures such as the 26-dimensional Leech lattice and the ‘Monster’, the largest sporadic finite group.

As for the physical or string theory motivation, one can argue that string theorists have been ‘speaking in the prose of Lorentzian algebras’ all along, because of the correspondence between the basic three-string vertex and a commutator of two vertex operators, giving a third one. Hence, the ‘structure constants’ of such an algebra should determine the form and strength of the three-string vertex, in conformity with Witten’s expectation [6] that E_{10} must be related to string field theory, presumably in ten dimensions. One may then further speculate that the lower rank, especially the $r = 4$, hyperbolic algebras may be related to string interactions in our four-dimensional spacetime.

In this context, an especially curious, but not commonly emphasised, fact is that Lorentzian algebras work even without latticising the momenta; all one needs is an integral value for the dot product of the two momenta entering the commutator (see [2]). But this is automatically ensured if the three momenta are on the mass-shell values allowed by the string spectrum! A different application of hyperbolic algebras to string theory has also been given recently by Kostelecky and Lechtenfeld [7], who use bosonised ghosts in place of Fubini–Veneziano fields on vertex operators.

The paper is organised as follows. In § 2, we summarise relevant algebraic notions and define Lorentzian, hyperbolic and strictly hyperbolic Dynkin diagrams. In § 3.1, after discussing the $r = 2$ case, we describe our method for the classification of hyperbolic Dynkin diagrams. Strictly hyperbolic cases are classified in § 3.2, followed by the hyperbolic ones in § 3.3. The latter subsection also contains a detailed proof of the impossibility of hyperbolic diagrams beyond rank 10. The results are displayed in tables 1 and 2. Section 4 ends the paper with a discussion mostly concerning Lorentzian algebras that are not of the hyperbolic type.

2. Mathematical results and definitions

In this section, we will describe the relationship between various algebraic concepts that will be needed for a classification of hyperbolic algebras. We shall assume the reader is familiar with simple roots, Cartan matrices and Dynkin diagrams for finite simple Lie algebras (FSLA) and condense the presentation accordingly. For the omitted details and definitions, we refer the reader to [1, 2].

A generalised Cartan matrix (GCM) is an indecomposable (i.e. not reducible to block-diagonal form by shuffling rows or columns) $r \times r$ matrix A_{ij} such that:

- (Ai) $A_{ij} \in \mathbb{Z}$
- (Aii) $A_{ii} = 2$
- (Aiii) $A_{ij} \leq 0 \quad i \neq j$
- (Aiv) $A_{ij} = 0 \Rightarrow A_{ji} = 0$.

To a given matrix \mathbf{A} , we may associate a Dynkin diagram (DD), denoted by $S(\mathbf{A})$ with the properties below.

- (Di) $S(\mathbf{A})$ has r vertices.
- (Dii) When $A_{ij}A_{ji} = n \leq 4$, the vertices i and j are joined by n lines.
- (Diii) If $|A_{ij}| > |A_{ji}|$, an arrow on the line (ij) points towards the vertex j .
- (Div) If $n > 4$, i and j are joined by a thick line on which we write $(|A_{ij}|, |A_{ji}|)$ with $|A_{ij}| \geq |A_{ji}|$. This $n > 4$ case will only concern us when we discuss rank-2 hyperbolic algebras.

An algebra of rank r can be defined for a given A_{ij} through

- (ai) $[h_i, h_j] = 0$
- (aia) $[h_i, e_j] = A_{ij}e_j$
- (aiia) $[h_i, f_j] = -A_{ij}f_j$
- (aiv) $[e_i, f_j] = \delta_{ij}h_j$

and the Serre relations†

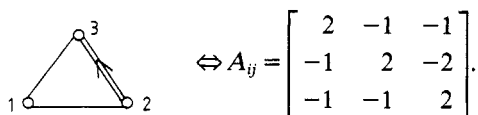
- (av) $ad(e_i)^{-A_{ij}+1}f_j = ad(f_i)^{-A_{ji}+1}e_j = 0$.

Thus the relationship between a GCM, a DD and an algebra as in (ai)-(av) is one-to-one.

The reader has undoubtedly recognised that the structure (ai)-(av) has been abstracted from a finite simple Lie algebra, where the h_i comprise the Cartan subalgebra (CAS), while e_i (f_i) is the raising (lowering) operator corresponding to the simple root α_i . In fact, for a FSLA, we have

$$A_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j}.$$

Hence for FSLA, the GCM, the DD and the simple root system carry exactly the same information (except for an irrelevant overall root length scale and orientation). The more general definition of A_{ij} in (Ai)-(Aiv) divorces the root system from the other concepts. With an arbitrary A_{ij} obeying (Ai)-(Aiv), we may define an algebra through (ai)-(av), even when a corresponding root space with a consistent inner product cannot be constructed. An example is provided by



Identifying this with $(2\alpha_i \cdot \alpha_j)/(\alpha_j \cdot \alpha_j)$ would lead to the inconsistency that α_3 is both shorter than and of equal length to α_2 , no matter the choice of the inner product!

In the rest of this paper we will exclude A_{ij} of the kind in the above paragraph through the criterion of *symmetrisability*, defined and employed from proposition 5.10 onwards in Kac [1]. A symmetrisable GCM \mathbf{A} may be written in the form $\mathbf{A} = \mathbf{D}\mathbf{G}$, with $D_{ij} = d_i\delta_{ij}$ and $G_{ij} = G_{ji}$. The entries of \mathbf{D} and \mathbf{G} are rational numbers in general. It is easy to see that the A_{ij} in the above example cannot be expressed in this form.

† This form of the commutation relations, while equivalent to the usual one in which commutators of any two roots are explicitly related to the other roots by structure constants, is not equally useful in practice.

It is now possible to think of the symmetric matrix G_{ij} as a metric on a root space and make the identifications

$$D_{ij} = \frac{2}{\alpha_i \cdot \alpha_i} \delta_{ij} \quad G_{ij} = \alpha_i \cdot \alpha_j.$$

This decomposition is of course unique only up to an overall scale factor as usual.

Given the class of symmetrisable GCM and their associated algebras, we still have different possibilities for the signature of the metric G_{ij} . We will only consider three cases.

- (i) The metric is positive definite with signature $(++ \dots +)$. This yields the FSLA.
- (ii) The metric is positive semi-definite with one null direction, i.e. the signature is $(++ \dots, 0)$. This results in affine Kac–Moody algebras (AKMA).
- (iii) The metric is Minkowskian with signature $(++ \dots +, -)$. This corresponds to Lorentzian algebras.

The Dynkin diagrams, root spaces and representation theory of the cases (i) and (ii) have been exhaustively classified and studied [1]. Here we will concentrate on two subclasses of case (iii). These are:

- (a) hyperbolic Dynkin diagrams (HDD), which revert to FSLA or AKMA Dynkin diagrams upon the deletion of any vertex;
- (b) strictly hyperbolic Dynkin diagrams (SHDD), which yield *only* FSLA Dynkin diagrams under the same operation.

It is not obvious that the HDD and the SHDD defined in this fashion correspond to a G_{ij} of signature $(++ \dots +, -)$; for this result we refer to Kac [1], p 47. Alternatively, one may arrive at the same conclusion by checking the signatures of the finite number of possible HDD and SHDD ($r > 2$) displayed in the next section.

3. Dynkin diagrams of hyperbolic algebras

3.1. The method

The $r = 2$ case has been treated by Lepowsky and Moody [8]. For completeness, we summarise the classification. We are automatically restricted to the strictly hyperbolic case as removal of one of the vertices leaves behind A_1 . The only type of diagram consistent with (Di)-(Div) and the definition of a SHDD is clearly $\circ \overset{kk'}{\text{---}} \circ$, with $kk' > 4$, $k \geq k' > 0$. Thus there are infinitely many SHDD for $r = 2$. The corresponding GCM is

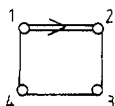
$$\mathbf{A} = \begin{bmatrix} 2 & -k \\ -k' & 2 \end{bmatrix} = \mathbf{D}\mathbf{G} = \begin{bmatrix} k/k' & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2k'/k & -k' \\ -k' & 2 \end{bmatrix}.$$

We see that \mathbf{A} is symmetrisable. It is also very easy to check that \mathbf{G} has signature $(+-)$ as expected.

In the next subsections we shall see in detail that there are no symmetrisable SHDD for $r > 4$, no symmetrisable HDD for $r > 10$ and that the total number of such algebras is finite (albeit over a hundred!), confirming results stated in [1]. At the high-rank end of the spectrum, $r = 7, 8, 9$ and 10 are given in Kac [1] and are 18 in number. The 118 cases between 3 and 6 constitute the chief content of the present study.

The general strategy in searching for hyperbolic Dynkin diagrams of rank $r + 1$ is as follows: (i) draw all possible Lie and/or affine (including semi-simple) diagrams of rank r ; (ii) add an extra root, trying all possible lengths; (iii) try connecting the

new root to the old ones in all the ways consistent with a symmetrisable GCM; for example, the diagram

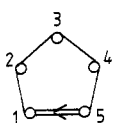


is not allowed since it inconsistently assigns both of the roots 3 and 4 two different lengths (we recall the convention that the arrow points from the long to the short root); (iv) test the resulting diagram by removing any point to see whether it reduces to (perhaps a disconnected combination of) known finite or affine algebras, the twisted ones being included among the latter. A diagram that survives the test is of the hyperbolic type.

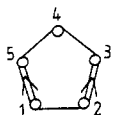
If such a case-by-case approach seems unsatisfactory to the reader, he should recall that the Dynkin diagrams (or equivalently, the Cartan matrices) of the five exceptional Lie algebras are found in a similar way, as the standard treatment in Jacobson [9] or Cahn [10] shows. It is no coincidence, but a reflection of the fact that the new algebras are also exceptional, that one has to resort to the same method here. In fact, the algebras with rank ≥ 7 are essentially built around the E series, while G_2 and F_4 play a prominent role for lower rank. We now apply the method and present the results in order of increasing rank, starting with the strictly hyperbolic cases.

3.2. Strictly hyperbolic algebras

It is not difficult to see that these stop at rank 4. Consider the possible diagrams with five points. Those with triangles or squares are ruled out, as the deletion of a point outside the closed subdiagram would give us a closed diagram which does not correspond to any finite Lie algebra. The next closed figure, a pentagon, represents $A_4^{(1)}$ if the roots are of the same length. The diagram

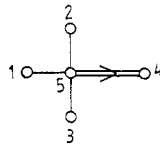


although leading to an 'acceptable' looking Cartan matrix if decoded naively, must be ruled out as it implies the roots 1, 2, 3 and 4 are long and short at the same time! A pentagon of the form

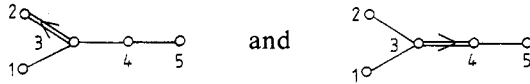


or its dual, yields an affine rather than a finite algebra when root 4 is removed. This leaves tree diagrams of three sorts: (a) a linear one, (b) one in a D_5 -type configuration, (c) one with four dots connected to a central point. Note that a G_2 triple line is forbidden as it would lead to $G_2^{(1)}$ or other non-Lie subdiagrams. Thus, only single and/or double lines may be used in (a), (b) and (c). The all-single-line cases are A_5 ,

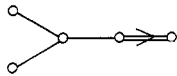
D_5 and $D_4^{(1)}$. In (c), we may use at most one double line to find



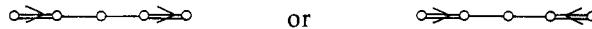
but removal of 1, 2 or 3 produces the affine algebra $B_3^{(1)}$. The dual version fails similarly. In (b) again only one double line is allowed. Of three such diagrams, the failures of



are evident upon dropping 5, while



is $B_4^{(1)}$. In the linear option (a), one may use two double lines only in the form (modulo duals)



but these are the affine algebras $A_8^{(2)}$ and $C_4^{(1)}$. Other attempts only produce $F_4^{(1)}$, $E_6^{(2)}$, B_5 or C_5 . The arguments establishing the absence of strictly hyperbolic algebras of higher rank proceed similarly, with certain simplifying restrictions such as the use of at most one double line per diagram and at most three branches per vertex. See also the proof of the absence of hyperbolic algebras for $r > 10$ in § 3.3, where more detail is provided.

The allowed strictly hyperbolic algebras of ranks 3 and 4 are displayed in table 1. The reader can easily check that these are the only possibilities by using the general method in § 3.1 and the restrictions mentioned subsequently. Let us briefly note a few interesting points.

(a) All of the diagrams of rank 3, except the first one, are based on the exceptional group G_2 . The unique rank-4 diagram can be thought of as an F_4 diagram closed upon itself, suggesting that the absence of strictly hyperbolic algebras of higher rank is related to the absence of suitable higher-rank exceptional algebras. $E_{6,7,8}$ are too large for the strictly hyperbolic case; however, they perform a similar function in the hyperbolic one as we shall see later.

(b) The diagrams 2, 4 and 5 for rank 3 indicate three different simple root lengths (from now on we will frequently omit the qualifier ‘simple’, since these will be the only roots we shall be concerned with), a situation not encountered in finite Lie algebras, although found in the twisted affine series $A_{2l}^{(2)}$.

3.3. Hyperbolic algebras

The proof that there are no hyperbolic algebras of rank > 10 proceeds along lines similar to the proof for strictly hyperbolic ones, except for the fact that we now allow

Table 1. Strictly hyperbolic algebras. In the subgroup column, the subgroup in the n th position from the left is obtained when the root numbered n is removed.

Rank	Number	G	G (dual)	Subgroups (dual subgroups)
3	1			B_2, A_2, B_2
	2			$G_2, A_1 \times A_1, G_2$
	3			G_2, A_2, G_2
	4			$G_2, A_1 \times A_1, B_2$
	5			$B_2, A_1 \times A_1, G_2$
4	1			C_3, B_3, B_3, C_3

affine (untwisted or twisted) algebra diagrams as possible subdiagrams. To show the impossibility of rank ≥ 11 , we need to establish a number of intermediate results. As we shall see, some of these rules will not apply for lower ranks, leading to a proliferation of diagrams for $r = 3, 4, 5$ and 6.

(I) All closed subdiagrams up to $(r - 2)$ -gons are forbidden, as the removal of one of the two outer points leaves us with a closed diagram (or diagrams) from which at least one root ‘dangles’. There are no such finite or affine algebras.

(II) An $(r - 1)$ -gon with a dangling root is also forbidden for $r > 10$. To see this, remove the sixth root from the junction; one is left with at best another hyperbolic algebra.

(III) $G_2, A_2^{(1)}$ and $A_2^{(2)}$ are obviously ruled out, as a subdiagram of at least four points containing them can always be isolated; such a diagram is not finite or affine.

(IV) We come to r -gons. With only one type of root, we have the affine algebra $A_{r-1}^{(1)}$. We must next rule out r -gons with double lines to be left with tree diagrams only.

(V) Double lines may not be introduced into an r -gon ($r \geq 11$) since it is always possible to delete a root so as to leave a double line in the middle rather than at the ends of a subdiagram. Again, there are no such finite or affine algebras.

We are now left with tree diagrams subject to the following restrictions which again hold for $r \geq 11$.

(VI) At most three branches may be connected to a vertex. Otherwise we could isolate a subdiagram of more than five points containing such a vertex. (Obviously this may be circumvented for $r=6$ since $D_4^{(1)}$ is of rank 5.)

(VII) There cannot be more than two vertices per diagram. Otherwise we could find and remove a point so as to end up with a connected diagram with two vertices and a branch of two more points beyond one of them.

(VIII) A double line may not be directly connected to a vertex as no such configuration is found among finite or affine diagrams.

(IX) Double lines may only be used at the ends of diagrams. This is again obvious, recalling the structure of Lie and affine diagrams of rank >5 .

(X) Four double lines are obviously ruled out, as are three double lines, albeit less obviously. A diagram with three double lines which are placed at three branch ends by (IX) must have a junction where these three branches meet. None of the double lines may be directly attached to the vertex by (VIII). Hence, removal of the point at the very end of one of the double lines results in an unacceptable diagram with two double lines and one 'protrusion' or dangling line left after the removal.

(XI) The above-mentioned subdiagram, with two double lines at the two ends of a branch and protrusions dangling from the same branch is also unacceptable as a hyperbolic diagram for $r \geq 11$. To see this, just remove the double line farthest away from the protrusion.

(XII) A linear diagram of equal-length roots with double lines added at both ends belongs to one of the finite or affine series $B_r, C_r, C_{r-1}^{(1)}, A_{2r}^{(2)}$ or $D_r^{(2)}$. Hence a hyperbolic diagram subject to all the previous restrictions can admit at most one double line at one end.

(XIII) By (VII), a diagram ending in a double line can have at most two vertices; but in fact for $r > 0$ it cannot have any. To rule out the two-vertex case, remove the root farthest from the double line. To rule out the single-vertex case, note that the vertex must be situated as far away from the double line as possible and must be either of the D_l or $E_{6,7,8}$ (shorter side) type. But the former diagram is just $B_{r-1}^{(1)}$, while removing the outermost point on the double line in the latter case gives something beyond $E_8^{(1)}$. This brings us back to (XII); thus double lines are out altogether!

(XIV) We are now restricted to single-line tree diagrams with at most three vertices. These can be only of two kinds: (i) one with two D_l type vertices, (ii) another one with one D_l and one $E_{6,7,8}$ type (shorter side) vertex. Other possibilities such as lengthening the E vertex or using two E vertices can be ruled out by removing the point beyond the E vertex or shortening one of the two E vertices. Now, (i) is already $D_{r-1}^{(1)}$, while deleting an extreme point from the E vertex leaves the case (ii) with something bigger than the $E_8^{(1)}$ diagram.

(XV) We are left with a single type of root, forming a Y-shaped diagram. Following Kac, we denote this by $T_{n,m,l}$ ($n \geq m \geq l$), where n, m, l refer to the lengths of the

branches including the vertex; thus $n + m + l = r + 2 \geq 13$. The combination $(n, 2, 2)$ is just D_r ; all others of the form $(n, m \geq 3, l \geq 2)$ can easily be checked to give diagrams beyond $E_8^{(1)}$ upon root deletions; hence this last class of candidates also cannot be hyperbolic. \square

A striking aspect of the above proof is again the special role played by the E series. It is clear that hyperbolic algebras stop at rank 10 essentially because the finite and affine E series stop at $E_8^{(1)}$; hyperbolic algebras of ranks 7, 8, 9, 10 exist because $E_{6,7,8}$ and $E_{6,7,8}^{(1)}$ exist. A glance at the high-rank end of table 2 will make this assertion clearer.

We can now refer the reader to table 2 where hyperbolic Dynkin diagrams of ranks between 3 and 10 are displayed in order of increasing rank. Within a given rank, the order roughly corresponds to increasing number of double, triple or quadruple lines and/or loops. Algebras with more than one root length sometimes have distinct dual partners obtained by interchanging root lengths, which is accomplished by changing the directions of the arrows; sometimes the algebra is self-dual. A very interesting point is the appearance of twisted and untwisted affine algebras as subalgebras of the same hyperbolic algebra. This is based on the fact that the duals of $B_r^{(1)}$, $C_r^{(1)}$, $F_4^{(1)}$ and $G_2^{(1)}$ are $A_{2r-1}^{(2)}$, $D_{r+1}^{(2)}$, $E_2^{(6)}$ and $D_4^{(3)}$. The twisted affine series $A_{2r}^{(2)}$ and the twisted algebra $A_2^{(2)}$ also make a number of hyperbolic algebras possible. The lesson is that the twisted Kac-Moody algebras are as indispensable as the finite or untwisted affine ones (and indeed, appear on an equal footing) in the construction of hyperbolic algebras.

$r=3$. The large number of rank-3 diagrams is due to the unique rank-2 diagrams representing $A_1^{(1)}$, $A_2^{(2)}$ and G_2 . These, in all possible combinations with an additional point, generate the rank-3 hyperbolic cases. The organisation of the table is based on the mentioned subdiagrams. Note the occurrence of the algebras (11-14, 17 and 18) with three different root lengths.

$r=4$. In the light of the arguments linking string theory in r dimensions to a hyperbolic algebra of the same rank, this class of diagrams (including especially the single strictly hyperbolic one) is of special interest. We note that $A_1^{(1)}$ and $A_2^{(2)}$, which are already affine, cannot appear as subdiagrams. G_2 , in contrast, still survives through its affinised forms $G_2^{(1)}$ and $D_4^{(3)}$. The first diagram is the lowest member of what is left of the A series in the hyperbolic case; this series extends to $r=9$, as we shall see. The determinant of the Cartan matrix of a rank- r member of this 'A series' is $-(r-1)$; in this sense these may be thought of as the groups 'su(- n)', $8 \geq n \geq 3$! Diagram 3 possesses an unusually high degree of symmetry, being invariant under the 24 operations of the permutation group P_4 . There are some diagrams of three root lengths after 11; in 12 we encounter for the first time a diagram of four different root lengths! The Cartan matrix for the algebra 16 has determinant -1 , a rare property it shares with E_{10} .

$r=5$. The algebra F_4 is what makes the cases 6, 8 and 9 possible. Diagrams 9 and 10 have three different root lengths.

$r=6$. Diagram 2 has appeared previously in [7]. Diagram 3 is strictly peculiar to rank 6, being based directly on $D_4^{(1)}$. The algebras 4, 5, 8, 9, 11 and 12 owe their existences to $F_4^{(1)}$ and its twisted dual $E_6^{(2)}$. Diagrams 12 and 13 involve three different root lengths.

Table 2. Hyperbolic algebras. In the subgroups column, the groups in parentheses in the n th position from the left are those obtained when the point number n is removed from the dual diagram.

Rank	Number	G	G (dual)	Subgroups (dual subgroups)
3	1			$A_2, A_1 \times A_1, A_1^{(1)}$
	2			$B_2, A_1 \times A_1, A_1^{(1)}$
	3			$A_1^{(1)}, A_1 \times A_1, A_1^{(1)}$
	4			$A_1^{(1)}, A_2, A_1^{(1)}$
	5			$A_2, A_1^{(1)}, A_2$
	6			$B_2, A_1^{(1)}, B_2$
	7			$A_1^{(1)}, A_1^{(1)}, A_1^{(1)}$
	8			$G_2, A_1 \times A_1, A_1^{(1)}$
	9			$G_2, A_1^{(1)}, G_2$
	10			$A_2, A_1 \times A_1, A_2^{(2)}$
	11			$B_2, A_1 \times A_1, A_2^{(2)}$
	12			$B_2, A_1 \times A_1, A_2^{(2)}$
	13			$G_2, A_1 \times A_1, A_2^{(2)}$
	14			$A_2^{(2)}, A_1 \times A_1, G_2$
	15			$A_2^{(2)}, A_1 \times A_1, A_2^{(2)}$
	16			$A_2^{(2)}, A_2, A_2^{(2)}$

Table 2. (continued)

Rank	Number	G	G (dual)	Subgroups (dual subgroups)
	17			$A_2^{(2)}, B_2, B_2$
	18			$A_1^{(1)}, A_1 \times A_1, A_2^{(2)}$
	19			$A_2^{(2)}, A_1^{(1)}, A_2^{(2)}$
4	1			$A_3, A_2 \times A_1, A_3, A_2^{(1)}$
	2			$A_2^{(2)}, A_2^{(1)}, A_3, A_3$
	3			$A_2^{(1)}, A_2^{(1)}, A_2^{(1)}, A_2^{(1)}$
	4			$C_2^{(1)}, (D_3^{(2)}), A_1 \times B_2, A_2 \times A_1, B_3(C_3)$
	5			$B_3(C_3), B_3(C_3), A_2 \times A_1, A_2^{(1)}$
	6			$C_2^{(1)}(D_3^{(2)}), B_3(C_3), A_3, B_3(C_3)$
	7			$C_2^{(1)}(D_3^{(2)}), C_2^{(1)}(D_3^{(2)}), C_2^{(1)}(D_3^{(2)}), A_1 \times A_1 \times A_1$
	8			$D_3^{(2)}, A_1 \times B_2, B_2 \times A_1, C_2^{(1)}$
	9			$C_2^{(1)}, D_3^{(2)}, D_2^{(1)}, D_3^{(2)}$
	10			$C_3(B_3), A_1 \times A_1 \times A_1, C_3(B_3), C_2^{(1)}(D_3^{(2)})$
	11			$A_4^{(2)}, A_1 \times B_2, A_1 \times A_2, B_3(C_3)$
	12			$A_4^{(2)}, A_1 \times B_2, A_1 \times B_2, A_4^{(2)}$

Table 2. (continued)

Rank	Number	G	G (dual)	Subgroups (dual subgroups)
	13			$A_4^{(2)}, A_1 \times B_2, B_2 \times A_1, D_3^{(2)}(C_2^{(1)})$
	14			$B_3, A_1 \times A_1 \times A_1, A_4^{(2)}, C_3$
	15			$A_4^{(2)}, A_1 \times A_1 \times A_1, C_2^{(1)}(D_3^{(2)}), A_4^{(2)}$
	16			$G_2^{(1)}(D_4^{(3)}), A_1 \times G_2, A_2 \times A_1, A_3$
	17			$D_4^{(3)}, A_1 \times A_2, A_2 \times A_1, G_3^{(1)}$
	18			$G_2^{(1)}(D_4^{(3)}), A_1 \times G_2, B_2 \times A_1, B_3(C_3)$
	19			$C_3(B_3), A_1 \times B_2, G_2 \times A_1, G_2^{(1)}(D_4^{(3)})$
	20			$G_2^{(1)}(D_4^{(3)}), G_2^{(1)}(D_4^{(3)}), A_1 \times A_1 \times A_1, A_3$
	21			$G_2^{(1)}(D_4^{(3)}), G_2^{(1)}(D_4^{(3)}), A_2 \times A_1, A_2^{(1)}$
	22			$G_2^{(1)}(D_4^{(3)}), A_1 \times G_2, A_1 \times G_2, G_2^{(1)}(D_4^{(3)})$
	23			$D_4^{(3)}, D_4^{(3)}, G_2^{(1)}, G_2^{(1)}$
5	1			$D_4, A_4, A_4, A_3 \times A_1, A_3^{(1)}$
	2			$D_4, A_3^{(1)}, D_4, A_3^{(1)}, A_3^{(1)}$
	3			$B_3^{(1)}(A_5^{(2)}), B_4(C_4), B_4(C_4), A_3 \times A_1, A_3^{(1)}$
	4			$B_3^{(1)}(A_5^{(2)}), A_1 \times B_3(A_1 \times C_3), A_2 \times A_1 \times A_1, B_4(C_4), A_4$
	5			$B_3^{(1)}(A_5^{(2)}), B_3^{(1)}(A_5^{(2)}), B_3^{(1)}(A_5^{(2)}), D_4, A_1 \times A_1 \times A_1 \times A_1$

Table 2. (continued)

Rank	Number	G	G (dual)	Subgroups (dual subgroups)
6				$F_4, F_4, A_1 \times A_1 \times A_2, A_3 \times A_1, B_3^{(1)}(A_5^{(2)})$
7				$B_3^{(1)}(A_5^{(2)}), A_1 \times B_3(C_3), B_2 \times A_1 \times A_1, B_4(C_4), D_4^{(2)}(C_3^{(1)})$
8				$F_4, C_3^{(1)}(D_4^{(2)}), F_4, B_4(C_4), B_4(C_4)$
9				$A_6^{(2)}, A_1 \times B_3(C_3), A_2 \times B_2, A_1 \times B_3(C_3), F_4$
10				$B_4(C_4), A_1 \times A_1 \times B_2, A_1 \times C_3(B_3), A_5^{(2)}(B_3^{(1)}), A_6^{(2)}$
6	1			$D_5, D_5, A_5, A_5, A_4 \times A_1, A_5^{(1)}$
	2			$D_5, D_5, D_5, A_1 \times A_1 \times A_1 \times A_2, D_4 \times A_1, D_4^{(1)}$
	3			$D_4^{(1)}, D_4^{(1)}, D_4^{(1)}, D_4^{(1)}, D_4^{(1)}, A_1 \times A_1 \times A_1 \times A_1 \times A_1$
	4			$E_6^{(2)}, A_1 \times C_4, A_2 \times A_3, A_3 \times A_2, B_4 \times A_1, F_4^{(1)}$
	5			$F_4^{(1)}(E_6^{(2)}), A_1 \times F_4, A_2 \times C_3(B_3), A_3 \times A_2, A_4 \times A_1, B_5(C_5)$
	6			$B_4^{(1)}(A_5^{(2)}), A_1 \times B_4(C_4), A_2 \times A_1 \times B_2, A_4 \times A_1, D_5, B_5(C_5)$
	7			$B_4^{(1)}(A_5^{(2)}), B_4^{(1)}(A_7^{(2)}), B_4^{(1)}(A_7^{(2)}), A_1 \times A_1 \times A_1 \times B_2, D_4 \times A_1, D_4^{(1)}$
	8			$F_4^{(1)}(E_6^{(2)}), F_4^{(1)}(E_6^{(2)}), A_1 \times A_1 \times C_3(B_3), A_3 \times A_2, D_4 \times A_1, B_4^{(1)}(A_7^{(2)})$
	9			$C_4^{(1)}(D_5^{(2)}), A_1 \times C_4(B_4), A_2 \times C_3(B_3), B_2 \times B_3(C_3), A_1 \times F_4, E_6^{(2)}(F_4^{(1)})$

Table 2. (continued)

Rank	Number	G	G (dual)	Subgroups (dual subgroups)
	10			$B_4^{(1)}(A_7^{(2)}), A_1 \times B_4(C_4), A_1 \times B_2 \times B_2, A_1 \times B_4(C_4), B_4^{(1)}(A_7^{(2)}), D_5^{(2)}(C_4^{(1)})$
	11			$E_6^{(2)}, C_4^{(1)}, E_6^{(2)}, F_4^{(1)}, D_5^{(2)}, F_4^{(1)}$
	12			$F_4^{(1)}(E_6^{(2)}), A_1 \times F_4, B_2 \times C_3(B_3), C_3 B_3, C_3(B_3) \times A_2, C_4(B_4) \times A_1, A_8^{(2)}$
	13			$B_4^{(1)}, A_1 \times B_4, B_2 \times A_1 \times B_2, C_4 \times A_1, A_7^{(2)}, A_8^{(2)}$
7	1			$E_6, D_6, D_6, A_7, A_7, A_5 \times A_1, A_5^{(1)}$
	2			$D_5^{(1)}, A_1 \times D_5, A_1 \times A_2 \times A_3, A_4 \times A_1 \times A_1, E_6, E_6$
	3			$B_5^{(1)}(A_9^{(2)}), A_1 \times B_5(C_5), A_2 \times A_1 \times B_3(C_3), A_4 \times B_2, D_5 \times A_1, E_6, B_6(C_6)$
8	1			$E_7, E_7, D_7, D_7, A_7, A_7, A_6 \times A_1, A_6^{(1)}$
	2			$E_7, E_7, A_1 \times A_6, A_1 \times A_6, A_1 \times A_2 \times A_3, A_5 \times A_2, E_6 \times A_1, E_6^{(1)}$
	3			$D_6^{(1)}, A_1 \times D_6, A_1 \times A_3 \times D_4, D_7, A_4 \times A_3, A_1 \times A_1 \times D_5, E_7, E_7$
	4			$B_6^{(1)}(A_{11}^{(2)}), A_1 \times B_6(C_6), A_3 \times A_1 \times B_4(C_4), B_7(C_7), A_4 \times B_3(C_3), D_5 \times B_2, E_6 \times A_1, E_7$
9	1			$E_7^{(1)}, E_8, E_8, D_8, D_8, A_8, A_8, A_6 \times A_1, A_8^{(1)}$
	2			$E_7^{(1)}, A_1 \times E_7, A_2 \times D_6, A_3 \times A_5, A_4 \times A_3 \times A_1, A_8, A_2 \times A_6, A_1 \times D_7, E_8$

Table 2. (continued)

Rank	Number	G	G (dual)	Subgroups (dual subgroups)
3				$D_7^{(1)}, A_1 \times D_7, A_1 \times A_2 \times D_5, D_8, A_4 \times D_4, A_3 \times D_5, A_1 \times A_1 \times E_6, E_8, E_8$
4				$B_7^{(1)}(A_{13}^{(2)}), A_1 \times B_7(C_7), A_1 \times A_2 \times B_5(C_5), B_8(C_8), A_4 \times B_4(C_4), D_5 \times B_3(C_3), E_6 \times B_2, A_1 \times E_7, E_8$
10	1			$E_8^{(1)}, A_1 \times E_8, A_2 \times E_7, A_3 \times E_6, A_4 \times D_5, A_5 \times A_4, A_1 \times A_2 \times A_6, A_9, A_1 \times A_8, D_9$
	2			$D_8^{(1)}, A_1 \times D_8, A_1 \times A_2 \times D_6, D_9, A_4 \times D_5, D_5 \times D_4, E_6 \times A_3, A_1 \times A_1 \times E_7, E_8^{(1)}, E_8^{(1)}$
	3			$B_8^{(1)}(A_{15}^{(2)}), A_1 \times B_8(C_8), A_1 \times A_2 \times B_6(C_6), B_9(C_9), A_4 \times B_5(C_5), D_5 \times B_4(C_4), E_6 \times B_3(C_3), E_7 \times B_2, E_8 \times A_1, E_8^{(1)}$

$r = 7$. All the diagrams for $r \geq 7$ have already been given by Kac [1]. Case 2 can be thought of as a contribution of the rank-6 diagram 2 or a cross between the E and the D series. Similarly, 3 is a cross between the E and B(C) series that started with 4 from rank 5 and 6 from rank 6. All algebras are based on E_6 .

$r = 8$. Diagram 2 is denoted by $T_{4,3,3}$ in [1]. It is the first example of its kind, while 1, 3 and 4 are higher-rank members of series we have previously seen. All diagrams are built upon the E series in its finite and affine forms. This will also hold for the remaining $r = 9$ and 10 cases.

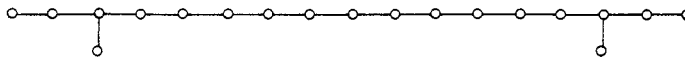
$r = 9$. Diagram 2 is $T_{5,4,2}$; otherwise everything is as for $r = 8$.

$r = 10$. The 'A series' has disappeared as predicted in (II), there being no affine algebra beyond $E_8^{(1)}$ to support it. The diagram 1 is $T_{7,3,2}$, alias the famous E_{10} , deserving the latter name both because of its rank and fact that its Cartan matrix A_{ij} has determinant -1 , in conformity with the general result $\det(A_{ij}[E_n]) = 9 - n$, where $E_9 = E_8^{(1)}$. Thus with E_{10} and the algebras represented by diagrams 2 and 3 the hyperbolic algebras come to an end.

4. Discussion

Or do they? While in the strict sense they do, Lorentzian extensions with roots still in a latticised Minkowski space are possible. These are distinct from the more restricted hyperbolic class in at least two important respects: (i) the Dynkin diagram no longer

has only finite or affine proper subdiagrams; (ii) the number of points on the diagram may, in general, be greater than the rank of the algebra. The second possibility, for which we will give an example, arises from the fact that a number of simple roots greater than the dimensionality of spacetime may be needed to reach all of the lattice points. Interesting examples of such diagrams have been discussed by Goddard and Olive [2] in the context of even self-dual Minkowskian lattices denoted by $\Pi^{8n+1,1}$. Here $(8n+1, 1)$ indicates the signature and the allowed dimensions of spaces in which such lattices can occur. The unique lattice for $n=1$ is known to correspond to the root lattice of E_{10} . That the Cartan matrix has determinant -1 and the simple roots are of equal length are essential to this result. One of the two possible diagrams given by Goddard and Olive for $n=2$ illustrates both novelties. This is the diagram



which can be thought of as two $E_8^{(1)}$ diagrams tied together by an additional point in the middle. This means 19 simple roots furnish a basis for this 18-dimensional lattice. Note that the Cartan matrix determinant for this diagram vanishes, reflecting the linear dependence of the 19 simple roots. The situation becomes extreme for $\Pi^{25,1}$; the Dynkin diagram is infinite [2]! Thus once again a 26-dimensional Minkowski spacetime emerges as a limiting case, just as it does for the unitarity and covariance of the bosonic string. We find it very intriguing that the numbers 26, 10 and 4 appear as upper bounds on the ranks of a special class of Lorentzian algebras, hyperbolic algebras and strictly hyperbolic algebras, respectively. Our conclusion is that a deeper understanding of such algebras will provide new insights into string theory and vice versa. Recent history supports this assertion: vertex operators, invented for dual model calculations, have proved to be ideal tools for the realisation of Kac-Moody and Lie algebras, whereas the vertex operator representation of Lie algebras with roots in self-dual lattices made the heterotic string possible.

Regarding the actual classification of Lorentzian diagrams, we are not aware of a systematic attempt along the direction of the present paper. However, a number of papers by Vinberg [11] examining the discrete groups in Lobachevski spaces may be relevant. Vinberg's basic approach[†] is to translate Coxeter's work [12] on the classification of discrete reflection groups for spheres S^n or Euclidean spaces E^n to hyperboloidal (Lobachevski) spaces Λ^n . As Coxeter's method leads to a classification of Lie algebra root configurations, this translation may very well provide a method for the description of Lorentzian root structures, although [11] does not directly address this problem.

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[†] The figures at the end of the first article of [11] are not to be interpreted as conventional Dynkin diagrams. For example, a list that appears superficially to resemble finite Lie algebra diagrams also includes the diagram for the affine algebra $G_2^{(1)}$, our hyperbolic algebra 16 of rank 4 and a G_2 -type diagram with m lines, with $m \geq 5$. All in all, the overlap (at the level of superficial similarity of diagrams) between our work and Vinberg's consists of 11 diagrams plus the ' E_{18} ' case discussed in the last section.

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